INEQUALITIES

UNIT 1 CLASSICAL INEQUALITIES

1. Inequality of the Means

To motivate our discussion, let's look at several situations.

(A) A man drove for 2 hours. In the first hour he travelled 16 km, and in the second hour he travelled 32 km. What is his average speed for the whole journey?

(B) A man drove from City P to City Q at a speed of 16 km/h and returned at 32 km/h. What is his average speed for the whole journey?

(C) In the traditional Chinese game mahjong, the value (in order not to 'encourage' gambling we use the term 'value' in place of 'payoff') of a hand is determined by the number of points of the hand and usually grows in a geometric sequence. For instance,

Points	0	1	2	3	4	5	6
Value	4	8	16	32	64	128	256

But in this case the value grows too fast. Some people want to modify it so that the value doubles every 2 points rather than every point. For instance,

Points	0	1	2	3	4	5	6
Value	4	x	8	y	16	z	32

What should be x, y and z? Some people tend to simply put $x = 6$, $y = 12$ and $z = 24$, but then the numbers in the second row will no longer form a geometric sequence. Can we choose x, y, z so that 4, x, 8, y, 16, z, 32 form a geometric sequence?

These questions should be fairly easy. For (A), the average speed is

$$\frac{16+32}{2} = 24 \text{ km/h}.$$

For (B), suppose the distance between P and Q is d km. Then the time taken for the journey from P to Q and the return journey are $\frac{d}{16}$ and $\frac{d}{32}$ hours, respectively. Hence the average speed is

$$\frac{2d}{\frac{d}{16}+\frac{d}{32}}=\frac{2}{\frac{1}{16}+\frac{1}{32}}=\frac{64}{3}\text{ km/h}.$$

For (C), in order that the numbers form a geometric sequence, we must have $\frac{x}{4}=\frac{8}{x}$, i.e. $x=\sqrt{4\cdot 8}=4\sqrt{2}$. Similarly, we get $y=\sqrt{8\cdot 16}=8\sqrt{2}$ and $z=\sqrt{16\cdot 32}=16\sqrt{2}$.

The above three situations illustrate the three 'means' that we are going to study. In (A), the 'mean' $\frac{16+32}{2}=24$, the 'usual' mean, is known as the **arithmetic mean** (A.M.). The mean in (B), $\frac{2}{\frac{1}{16}+\frac{1}{32}}=\frac{64}{3}$, is known as the **harmonic mean** (H.M.). The mean in (C), $\sqrt{16\cdot 32}=16\sqrt{2}$, is known as the **geometric mean** (G.M.).

In general, given n positive numbers a_1, $a_2,\ldots,$ a_n we define the three means as follows:

$$\begin{aligned}\text{A.M.}&=\frac{a_1+a_2+\cdots+a_n}{n}\\ \text{G.M.}&=\sqrt[n]{a_1a_2\cdots a_n}\\ \text{H.M.}&=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}\end{aligned}$$

From the previous three examples, we see that for the particular case $n=2$, $a_1=16$ and $a_2=32$, A.M. ≥ G.M. ≥ H.M.. Indeed, for $n=2$ this is not hard to prove:

$$\begin{aligned}\frac{2}{\frac{1}{a_1}+\frac{1}{a_2}}=\frac{2a_1a_2}{a_1+a_2}&=\frac{2a_1a_2}{\left(\sqrt{a_1}-\sqrt{a_2}\right)^2+2\sqrt{a_1a_2}}\le\frac{2a_1a_2}{2\sqrt{a_1a_2}}\\ &=\sqrt{a_1a_2}\\ &=\sqrt{\frac{\left(a_1+a_2\right)^2-\left(a_1-a_2\right)^2}{4}}\le\sqrt{\frac{\left(a_1+a_2\right)^2}{4}}\\ &=\frac{a_1+a_2}{2}\end{aligned}$$

In fact, this is true for the n-variable case in general. We have

Theorem 1.1. (AM-GM-HM inequality)

Let a_1, a_2, …, a_n be positive numbers. Then A.M. ≥ G.M. ≥ H.M. for these numbers, i.e.

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1a_2\cdots a_n} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Example 1.1.

Prove that for positive real numbers *a*, *b*, *c*,

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9.$$

Solution.

By the AM-HM inequality,

$$\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

Hence $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$.

Example 1.2.

Let *a* and *b* be real numbers such that $a > b > 0$. Determine the least possible value of

$$a + \frac{1}{b(a-b)}.$$

Solution.

By the AM-GM inequality,

$$a + \frac{1}{b(a-b)} = (a-b) + b + \frac{1}{b(a-b)} \geq 3\sqrt[3]{(a-b)\cdot b \cdot \frac{1}{b(a-b)}} = 3.$$

Equality holds if $a - b = b = \frac{1}{b(a-b)}$, i.e. $a = 2$ and $b = 1$.

Hence the answer is 3.

2. The Cauchy-Schwarz Inequality

Before formally presenting the inequality, let's consider an example.

Example 2.1.

Let *a*, *b*, *c*, *d* be real numbers. Prove that $(a^2+c^2)(b^2+d^2) \geq (ab+cd)^2$.

Determine when equality holds.

Solution.

If any one of *a*, *b*, *c*, *d* is zero, the inequality is trivial.

In particular, if $a=b=0$ or $c=d=0$, then equality holds.

Assume *a*, *b*, *c*, *d* are non-zero.

Then a^2d^2 and b^2c^2 are positive. By the AM-GM inequality, we have

$$a^2d^2 + b^2c^2 \geq 2\sqrt{(a^2d^2)(b^2c^2)} = 2|abcd| \geq 2abcd .$$

Adding $a^2b^2+c^2d^2$ on both sides, we have

$$a^2b^2 + c^2d^2 + a^2d^2 + b^2c^2 \geq a^2b^2 + c^2d^2 + 2abcd ,$$

i.e.

$$(a^2+c^2)(b^2+d^2) \geq (ab+cd)^2 .$$

Equality holds if and only if $a^2d^2 = b^2c^2$ and $|abcd| = abcd$, i.e. $ad = bc$.

Combining all cases, equality holds if and only if $ad = bc$.

Basically, the above inequality says that 'the product of the sum of squares is greater than the square of the sum of products'. Indeed, this generalizes to 2*n* variables. We have

Theorem 2.1. (Cauchy-Schwarz inequality)

Let a_1, a_2, …, a_n, b_1, b_2, …, b_n be real numbers. Then

$$(a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2)(b_1{}^2 + b_2{}^2 + \cdots + b_n{}^2) \geq (a_1b_1 + a_1b_2 + \cdots + a_nb_n)^2 .$$

Equality holds if and only if $a_ib_j = a_jb_i$ for all i, *j*, i.e.

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} \text{ or } b_1 = b_2 = \cdots = b_n = 0 .$$

Illustration. Take $a_1 = -2$, $a_2 = 3$, $a_3 = 5$, $b_1 = 1$, $b_2 = 0$, $b_3 = 4$. Then

$$(a_1^2 + a_2^2 + a_3^3)(b_1^2 + b_2^2 + b_3^2) = 646$$
$$(a_1b_1 + a_2b_2 + a_3b_3)^2 = 324$$

Remark 1. An easy way to memorise the Cauchy-Schwarz inequality is

$$\sum a^2 \sum b^2 \geq \left(\sum ab\right)^2$$

which actually means

$$\sum_{i=1}^{n} a_i^{\,2} \sum_{i=1}^{n} b_i^{\,2} \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2 .$$

Remark 2. By a slight modification, the Cauchy-Schwarz inequality holds for complex numbers as well. In fact, if a_1, a_2, …, a_n, b_1, b_2, …, b_n are complex numbers, then

$$\left(|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2\right)\left(|b_1|^2 + |b_2|^2 + \cdots + |b_n|^2\right) \geq |a_1b_1 + a_1b_2 + \cdots + a_nb_n|^2 .$$

Equality holds if and only if $a_ib_j = a_jb_i$ for all i, *j*.

Example 2.2.

Prove that for positive real numbers *a*, *b*, *c*,

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9 .$$

Solutions.

Note that this is the same inequality as in Example 1.1.

Here we apply the Cauchy-Schwarz inequality. We have

$$\begin{aligned}(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) &= \left[\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 + \left(\sqrt{c}\right)^2\right]\left[\left(\frac{1}{\sqrt{a}}\right)^2 + \left(\frac{1}{\sqrt{b}}\right)^2 + \left(\frac{1}{\sqrt{c}}\right)^2\right] \\ &\geq \left(\sqrt{a} \cdot \frac{1}{\sqrt{a}} + \sqrt{b} \cdot \frac{1}{\sqrt{b}} + \sqrt{c} \cdot \frac{1}{\sqrt{c}}\right)^2 \\ &= 9\end{aligned}$$

Remark. Equality holds if and only of $\frac{\sqrt{a}}{\frac{1}{\sqrt{a}}} = \frac{\sqrt{b}}{\frac{1}{\sqrt{b}}} = \frac{\sqrt{c}}{\frac{1}{\sqrt{c}}}$, i.e. $a = b = c$.

Example 2.3.

Let *x* and *y* be real numbers such that $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$.

Prove that $x^2 + y^2 = 1$.

Solution.

By the Cauchy-Schwarz inequality (with $a_1 = x$, $a_2 = \sqrt{1-x^2}$, $b_1 = \sqrt{1-y^2}$, $b_2 = y$), we have

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} \le \sqrt{\left[x^2 + \left(\sqrt{1-x^2}\right)^2\right]\left[y^2 + \left(\sqrt{1-y^2}\right)^2\right]} = 1.$$

Equality holds if and only if

$$\frac{x}{\sqrt{1-y^2}} = \frac{\sqrt{1-x^2}}{y},$$

i.e. $x^2y^2 = (1-x^2)(1-y^2)$, or $x^2 + y^2 = 1$.

Now $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$, so we must have $x^2 + y^2 = 1$.

The Cauchy-Schwarz inequality is unique in its special condition for equality case. Most other inequalities have equality case when the variables are equal. Indeed, observing the equality case is one important technique in proving inequalities, as we shall see in Unit 2.

3. The rearrangement inequality

A class of 40 students is split into four groups, *A*, *B*, *C* and *D*, with 7, 8, 12 and 13 students respectively. Each group is to choose a type of shirts to represent their group. These are four types of shirts, and their unit prices are \$10, \$15, \$20 and \$25 respectively. Of course, each student will buy a shirt representing his group and each group will choose a different type of shirts. What matching between groups and shirts will yield a minimum total expenses of the class on shirts? What matching will yield a maximum?

It is quite intuitive that the total expenses will be minimum when Group *D*, with the largest number of students, chooses the least expensive type of shirts, Group *C* chooses the second least expensive type of shirts, and so on. By contrast, maximality occurs when Group *D* chooses the most expensive type of shirts, Group *C* the second most expensive type of shirts and so on.

The above situation illustrates the following:

Theorem 3.1. (Rearrangement Inequality)

If a_i and b_i (i = 1, 2, …, n) are real numbers such that

$$a_1 \ge a_2 \ge \cdots \ge a_n \text{ and } b_1 \ge b_2 \ge \cdots \ge b_n,$$

then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \ge a_1b_{j1} + a_2b_{j2} + \cdots + a_nb_{jn} \ge a_1b_n + a_2b_{n-1} + \cdots + a_nb_1,$$

where $\{j_1, j_2, \ldots, j_n\}$ is a permutation of $\{1, 2, \ldots, n\}$. In other words,

Direct Sum ≥ Random Sum ≥ Reverse Sum.

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

Illustration. In the above example, $a_1 = 13$, $a_2 = 12$, $a_3 = 8$, $a_4 = 7$, $b_1 = 25$, $b_2 = 20$, $b_3 = 15$, $b_4 = 10$. We have

$$\begin{aligned} \text{Direct sum} &= 13(25) + 12(20) + 8(15) + 7(10) = 755 \\ \text{A random sum} &= 13(20) + 12(15) + 8(25) + 7(10) = 710 \\ \text{Reverse sum} &= 13(10) + 12(15) + 8(20) + 7(25) = 645 \end{aligned}$$

Example 3.1.

Let a_1, a_2, …, a_n be positive real numbers and $\{b_1, b_2, \ldots, b_n\}$ be a permutation of $\{a_1, a_2, \ldots, a_n\}$. Prove that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \ge n.$$

Solution.

Without loss of generality, assume $a_1 \ge a_2 \ge \cdots \ge a_n$. Then

$$\frac{1}{a_n} \ge \frac{1}{a_{n-1}} \ge \cdots \ge \frac{1}{a_1}.$$

By the fact that Random Sum ≥ Reverse Sum, we have

$$\begin{aligned}\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_n}{b_n} &= a_1\left(\frac{1}{b_1}\right)+a_2\left(\frac{1}{b_2}\right)+\cdots+a_n\left(\frac{1}{b_n}\right)\\ &\geq a_1\left(\frac{1}{a_1}\right)+a_2\left(\frac{1}{a_2}\right)+\cdots+a_n\left(\frac{1}{a_n}\right)\\ &= n\end{aligned}$$

as desired.

4. Other Inequalities

In this section we will look at generalizations and variations to the three inequalities we have learned.

We first look at two generalizations of the AM-GM inequality. For any real number s and positive real numbers $a_1, a_2, \ldots, a_n$, we define

$$M_s = \left(\frac{a_1^s + a_2^s + \cdots + a_n^s}{n}\right)^{\frac{1}{s}}$$

to be the ***s*-power mean** of $a_1, a_2, \ldots, a_n$. It is clear from this definition that M_1 is the familiar AM and M_{-1} is the HM. A little knowledge in calculus would enable us to define M_0 as $\lim_{s\to 0} M_s = (a_1 a_2 \cdots a_n)^{1/n}$ which is the GM. Hence Theorem 1.1 says $M_1 \geq M_0 \geq M_{-1}$. The following theorem generalizes this to any M_s.

Theorem 4.1. (Power Mean Inequality)
Let $a_1, a_2, \ldots, a_n$ be positive real numbers and *s, t* be real numbers such that $s > t$. Then

$$M_s \geq M_t.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Remark. The number

$$M_2 = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}$$

is usually known as the **root mean square** (RMS) or **quadratic mean** (QM) of a_1, a_2, …, a_n. Also, it is customary to define

$$M_{+\infty} = \lim_{s\to+\infty} M_s = \max\{a_1, a_2, ..., a_n\}$$
$$M_{-\infty} = \lim_{s\to-\infty} M_s = \min\{a_1, a_2, ..., a_n\}$$

This implies that all ‘means’ lie between the maximum and minimum of the group of numbers.

Example 4.1.

Given a set of data x_1, x_2, …, x_n with mean (the usual arithmetic mean) $\bar{x}$, we define the mean deviation $\bar{d}$ and the **standard deviation** σ as

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|$$
$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Show that $\bar{d} \le \sigma$.

Solution.

Consider the n non-negative numbers $|x_1 - \bar{x}|$, $|x_2 - \bar{x}|$, …, $|x_n - \bar{x}|$.

If all of these are zeros the result is trivial.

So we assume that exactly k of these numbers are non-zero, and without loss of generality we may assume that the first k of them are non-zero.

Applying the RMS-AM inequality ($M_2 \ge M_1$) on these k numbers, we have

$$\frac{1}{k}\sum_{i=1}^{n}|x_i - \bar{x}| \le \sqrt{\frac{1}{k}\sum_{i=1}^{n}|x_i - \bar{x}|^2}.$$

Note that $\frac{k}{n} \le 1$, so $\frac{k}{n} \le \sqrt{\frac{k}{n}}$. Multiplying this to the above inequality, we get

$$\frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}| \le \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2},$$

i.e.

$$\bar{d} \le \sigma.$$

Again let a_1, a_2, …, a_n be positive real numbers. We define P_k to be the average of all products of k of the a_i's, i.e.

$$\begin{aligned}
P_1 &= \frac{a_1 + a_2 + \cdots + a_n}{n} \\
P_2 &= \frac{a_1a_2 + a_1a_3 + \cdots + a_1a_n + a_2a_3 + \cdots + a_{n-1}a_n}{\frac{1}{2}n(n-1)} \\
&\vdots \\
P_n &= a_1a_2\cdots a_n
\end{aligned}$$

Hence the AM is simply P_1 and the GM is $P_n^{1/n}$. Theorem 1.1 says $P_1 \geq P_n^{1/n}$. The following theorem generalizes this.

Theorem 4.2. (Maclaurin's Symmetric Mean Inequality)
For positive real numbers a_1, a_2, …, a_n,

$$P_1 \geq P_2^{1/2} \geq \cdots \geq P_n^{1/n}.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Example 4.2.

Prove that for positive real numbers a, b, c,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a^8 + b^8 + c^8}{a^3b^3c^3}$$

Solution.

Since $M_8 \geq M_1$, we have $\left(\dfrac{a^8 + b^8 + c^8}{3}\right)^{\frac{1}{8}} \geq \dfrac{a+b+c}{3}$.

By the symmetric mean inequality,

$$P_1^8 = P_1^6P_1^2 \geq \left(P_3^{1/3}\right)^6\left(P_2^{1/2}\right)^2,$$

i.e.

$$\left(\frac{a+b+c}{3}\right)^8 \geq \frac{(abc)^2(ab+bc+ca)}{3}.$$

Combining these results, we have

$$\frac{a^8+b^8+c^8}{a^3b^3c^3} \geq 3\left(\frac{a+b+c}{3}\right)^8 \frac{1}{(abc)^3} \geq \frac{(abc)^2(ab+bc+ca)}{(abc)^3} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Now we proceed to look at a generalization of the Cauchy-Schwarz inequality.

Theorem 4.3. (Hölder's Inequality)

Let a_1, a_2, …, a_n, b_1, b_2, …, b_n be complex numbers, *p*, *q* be real numbers greater than 1 such that $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\left|a_1b_1+a_1b_2+\cdots+a_nb_n\right| \le \left(\left|a_1\right|^p+\left|a_2\right|^p+\cdots\left|a_n\right|^p\right)^{\frac{1}{p}}\left(\left|b_1\right|^q+\left|b_2\right|^q+\cdots\left|b_n\right|^q\right)^{\frac{1}{q}}.$$

Equality holds if and only if $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}$.

Remark. The Cauchy-Schwarz inequality is a special case of Hölder's inequality with $p=q=2$.

Example 4.3.

Prove that for positive real numbers *a*, *b*, *c*,

$$a^3+b^3+c^3 \ge a^2b+b^2c+c^2a.$$

Solution.

Setting $p=\frac{3}{2}$, $q=3$ in Hölder's inequality, we have

$$\begin{aligned} a^2b+b^2c+c^2a &\le \left[\left(a^2\right)^{\frac{3}{2}}+\left(b^2\right)^{\frac{3}{2}}+\left(c^2\right)^{\frac{3}{2}}\right]^{\frac{2}{3}}\left(a^3+b^3+c^3\right)^{\frac{1}{3}} \\ &= a^3+b^3+c^3 \end{aligned}$$

Alternative Solution.

Without loss of generality, assume $a \ge b \ge c$. Then $a^2 \ge b^2 \ge c^2$.

Since Direct Sum ≥ Random Sum, we have

$$a^2(a)+b^2(b)+c^2(c) \ge a^2(b)+b^2(c)+c^2(a),$$

i.e.

$$a^3+b^3+c^3 \ge a^2b+b^2c+c^2a.$$

Finally, we will look at a variation of the rearrangement inequality. The rearrangement inequality says

Direct Sum ≥ Random Sum ≥ Reverse Sum.

Now instead of the random sum, we put another expression in the middle. This is given by the following theorem.

Theorem 4.4. (Chebyschev's Inequality)

Let a_1, a_2, …, a_n, b_1, b_2, …, b_n be real numbers, $a_1 \geq a_2 \geq \cdots \geq a_n$, $b_1 \geq b_2 \geq \cdots \geq b_n$. Then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq \frac{(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)}{n} \geq a_1b_n + a_2b_{n-1} + \cdots + a_nb_1 .$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

Example 4.4.

Let a_1, a_2, …, a_n be positive real number less than 1, $a_1 + a_2 + \cdots + a_n = S$. Prove that

$$\frac{a_1}{1-a_1} + \frac{a_2}{1-a_2} + \cdots + \frac{a_n}{1-a_n} \geq \frac{nS}{n-S} .$$

Solution.

Without loss of generality assume $a_1 \geq a_2 \geq \cdots \geq a_n$.

Then $1-a_n \geq 1-a_{n-1} \geq \cdots \geq 1-a_1$ and so

$$\frac{a_1}{1-a_1} \geq \frac{a_2}{1-a_2} \geq \cdots \geq \frac{a_n}{1-a_n} .$$

By Chebyschev's inequality,

$$\begin{aligned}
S &= a_1 + a_2 + \cdots + a_n \\
&= \frac{a_1}{1-a_1}(1-a_1) + \frac{a_2}{1-a_2}(1-a_2) + \cdots + \frac{a_n}{1-a_n}(1-a_n) \\
&\leq \frac{1}{n}\left[\frac{a_1}{1-a_1} + \frac{a_2}{1-a_2} + \cdots + \frac{a_n}{1-a_n}\right]\left[(1-a_1) + (1-a_2) + \cdots + (1-a_n)\right] \\
&= \frac{1}{n}\left[\frac{a_1}{1-a_1} + \frac{a_2}{1-a_2} + \cdots + \frac{a_n}{1-a_n}\right](n-S)
\end{aligned}$$

Hence

$$\frac{a_1}{1-a_1}+\frac{a_2}{1-a_2}+\cdots+\frac{a_n}{1-a_n}\geq\frac{nS}{n-S}.$$

5. Exercises

In the following, all variables denote positive real numbers.

1. Prove that if

$$(1+a_1)(1+a_2)\cdots(1+a_n)=2^n,$$

then

$$a_1a_2\cdots a_n\le 1.$$

2. Prove that

$$\frac{{a_1}^2}{a_2}+\frac{{a_2}^2}{a_3}+\cdots+\frac{{a_n}^2}{a_1}\ge a_1+a_2+\cdots+a_n.$$

3. Prove that if *a*, *b*, *c* are less than 1 and $a+b+c=2$, then

$$\frac{abc}{(1-a)(1-b)(1-c)}\ge 8.$$

4. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}.$$

5. (IMO 1978) Let a_1, a_2, …, a_n be distinct positive integers. Prove that

$$1+\frac{1}{2}+\cdots+\frac{1}{n}\le a_1+\frac{a_2}{2^2}+\cdots+\frac{a_n}{n^2}.$$